STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Oxley College

TRIAL EXAMINATION 1996

4 UNIT MATHEMATICS

Time Allowed: 3 hours (Plus 5 minutes reading time)

Directions to Candidates

Attempt **all** questions.
All questions are of equal value.
All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
Standard integrals are printed on the last page.
Board approved calculators may be used.
Each question attempted should be labelled carefully and each question should be started on a new page with Examination number clearly visible.
If required additional writing paper may be obtained from the supervisors.

Question 1:

(a) Integrate (i) $\int \tan^3 x dx$ (3 marks)

(ii) $\int \frac{e^x + e^{2x}}{1+e^{2x}} dx$ (3 marks)

(b) Evaluate (i) $\int_{-1}^{1} \frac{dx}{x^2 + 2x + 5}$ (2 marks)

(ii) $\int_{4}^{12} \frac{dx}{(4+x)\sqrt{x}}$ (3 marks)

(c) An ellipse has its centre at the origin and its foci on the x axis. The distance between its foci is four units and the distance between its directrices is sixteen units. Find the cartesian equation of the ellipse. (3 marks)

Question 2:

(a) If z is a complex number

(i) Find the curve in the Argand plane for which $\text{Re}(z^2) = 3$. Sketch (2 marks)

(ii) Find the curve such that $\text{Im}(z^2) = 4$. Sketch. (2 marks)

(iii) Sketch the region defined by the intersection of the inequalities $0 < \text{Re}(z^2) < 3$ and $0 < \text{Im}(z^2) < 4$ (3 marks)

(b) Evaluate $\int_{0}^{\frac{\pi}{3}} \frac{dx}{1 - \sin x} dx$ (3 marks)

(c) Two of the zeros of the polynomial $P(x) = x^4 + bx^3 + cx^2 + dx + e$ $(b, c, d, e \in \Re)$ are $2 + i$ and $1 - 3i$. Find the other two zeros and hence find b and e. (4 marks)

Question 3:

(a) Show that $\operatorname{cosec} 2\theta + \cot 2\theta = \cot\theta$ for all $\theta \in \Re$. Hence find the exact value of $\cot\frac{\pi}{8}$.

Show that $\operatorname{cosec}\frac{2\pi}{15} + \operatorname{cosec}\frac{4\pi}{15} + \operatorname{cosec}\frac{8\pi}{15} + \operatorname{cosec}\frac{16\pi}{15} = 0$ (4 marks)

(b) The function $f(x)$ is given by $f(x) = x(x-2)^2(x+1)^3$.
On separate axes, sketch: (10 marks)

(i) $f(x)$ (ii) $y^2 = f(x)$ (iii) $y = f|x|$

(iv) $\frac{f(x)}{|x|}$ (v) $e^{f(x)}$

Question 4:

(a) Show that the ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ meet at right angles. (4 marks)

(b) Find the five real values of c for which $z = (1 + ic)^6$ is real. (4 marks)

(c) $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$

Show that the normal at P cuts the hyperbola again at $Q\left(\frac{-c}{t^3}, -ct^3\right)$. (6 marks)

Question 5:

(a) Find $\int e^x \sin x dx$ (3marks)

(b) Sketch the function $f(x) = |x-3| + |x+1|$.
Hence or otherwise solve the inequality $|x-3| + |x+1| > 6$ (3 marks)

(c) If $f(x) = \ln(\sec x + \tan x)$

(i) Show that $f'(x) = \sec x$. (1 mark)

(ii) Using (i) above and the substitution $x = a\tan\theta$, or otherwise, show that
$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left(x + \sqrt{a^2 + x^2}\right) + c.$ (3 marks)

(d) Given $P(x) = x^4 + x^3 - 3x^2 - 5x - 2 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(i) Find the equation with roots one more than the reciprocal of the original roots. (2 marks)

(ii) If $P(x)$ has a root of multiplicity 3, find all the roots of $P(x)$. (2 marks)

Question 6:

(a) (i) Show that $\omega = cis\frac{\pi}{7}$ is a zero of $P(z) = z^7 + 1$ (1 mark)

(ii) Show that, for all odd integers n, ω^n is a zero of $P(z)$ (1 mark)

(iii) If $P(z) = (z+1)Q(z)$, find $Q(z)$ and hence show that ω is a zero of $Q(z)$. (2 marks)

(iv) Find all the zeros of $Q(z)$ in mod-arg form. (2 marks)

(b) The curves
$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$
are called the hyperbolic trig functions.

(i) Show that $\cosh^2 x - \sinh^2 x = 1$. (1 mark)

(ii) Sketch $\cosh x$ and $\sinh x$ on the same axes. (3 marks)

(iii) The region between the curves and bounded by the y axis and the line $x = 1$ is rotated about the y axis. Show that the volume of the solid generated can be given by

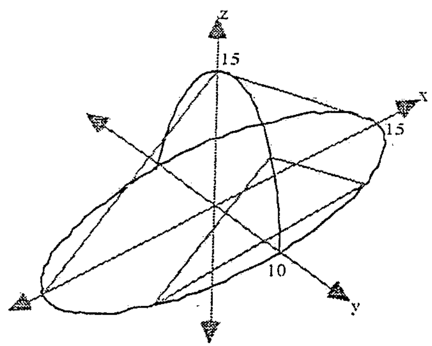
$$V = 2\pi\int_0^1 xe^{-x}dx$$

and hence calculate this volume. (4 marks)

Question 7:

(a) (7 marks)

For Kyle's birthday, and her great love of conics, her friends Jonno and Christo make her a Bowral Mudcake, with its base in the shape of an ellipse, with semi-major axis 15 cm and semi-minor axis 10 cm. Slices of the cake are isosceles triangles, whose vertices trace out a semi-elliptical path with the same semi-major axis and semi-minor axis lengths as the base. By taking suitable slices, find the volume of Kyle's birthday cake.



(b) Find $\int \frac{x^3}{(x^2+1)^3}dx$ using the substitutions (i) $u = x^2 + 1$

(ii) $x = \tan\theta$

Explain the difference in results algebraically. (7 marks)

Question 8:

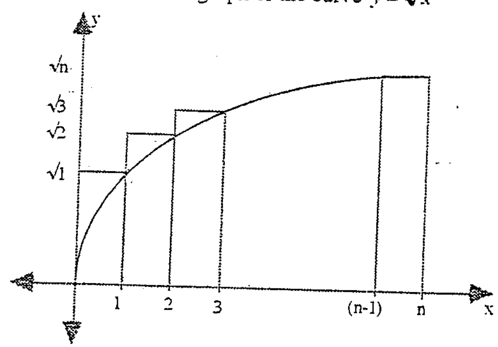
(a) The vertices of a quadrilateral ABCD lie on the circumference of a circle of radius r units.

The angles subtended at the centre by the sides AB, BC, CD, DA respectively are in arithmetic sequence with first term α and common difference β.

(i) Show that $2\alpha + 3\beta = \pi$ and interpret this result geometrically. (2 marks)

(ii) Show that the area A of ABCD is given by $A = r^2(\sin\alpha + \sin(\alpha+\beta))$ (4 marks)

(b) Consider the graph of the curve $y = \sqrt{x}$



(i) Show that $\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \ldots\ldots + \sqrt{n} \geq \int_0^n \sqrt{x}dx = \frac{2}{3}n\sqrt{n}$. (2 marks)

(ii) Show that $(4k+3)\sqrt{k} < (4k+1)\sqrt{k+1}$ (for k a positive integer) (2 marks)

(iii) Use Mathematical Induction and (ii) to show that
$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \ldots\ldots + \sqrt{n} \leq \frac{4n+3}{6}\sqrt{n}.$$ (3 marks)

(iv) Hence evaluate $\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \ldots\ldots + \sqrt{10000}$ to the nearest 100 (1 mark)

1996 4unit TRIAL SOLUTIONS

1(a)(i) $\int \tan^3 x\,dx$

$= \int (\sec^2 x - 1)\tan x\,dx$

$= \int \sec^2 x \tan x\,dx - \int \frac{\sin x}{\cos x}\,dx$

$= \frac{1}{2}\tan^2 x + \ln\cos x + c$

(ii) $\int \frac{e^x + e^{2x}}{1+e^{2x}}\,dx$

$= \int \frac{e^x\,dx}{1+e^{2x}} + \int \frac{e^{2x}\,dx}{1+e^{2x}}$

$= \tan^{-1} e^x + \frac{1}{2}\ln(1+e^{2x}) + c$

(b)(i) $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

$= \int_{-1}^{1} \frac{dx}{(x+1)^2+4} = \frac{1}{2}\left[\tan^{-1}\frac{x+1}{2}\right]_{-1}^{1}$

$= \frac{\pi}{4}\cdot\frac{1}{2} = \frac{\pi}{8}$

(ii) $\int_4^{12} \frac{dx}{(4+x)\sqrt{x}}$

$= \int_2^{2\sqrt{3}} \frac{2\,du}{4+u^2}$

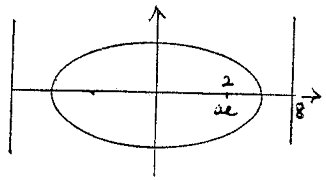
$u = \sqrt{x}$
$du = \frac{1}{2\sqrt{x}}\,dx$
$x = 4 \quad u = 2$
$x = 12 \quad u = 2\sqrt{3}$

$= \left[\tan^{-1}\frac{u}{2}\right]_2^{2\sqrt{3}}$

$= (\tan^{-1}\sqrt{3} - \tan^{-1} 1)$

$= \left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{12}$

(c)



$\frac{a}{e} = 8$, $ae = 2$ $\left(e = \frac{2}{a}\right)$

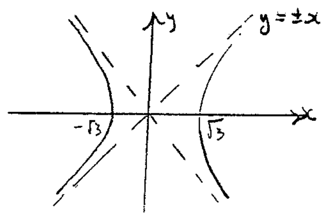
$\therefore a^2 = 16 \quad a = 4$
$e = \frac{1}{2}$

$\therefore b^2 = 16\left(1 - \frac{1}{4}\right) = 12$

ie $\frac{x^2}{16} + \frac{y^2}{12} = 1$ is the eqn.

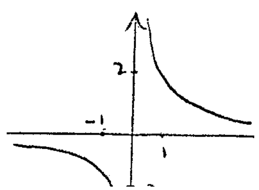
2 (i) $z^2 = x^2 + 2xyi - y^2$

ie $x^2 - y^2 = 3$ is $Re(z^2) = 3$

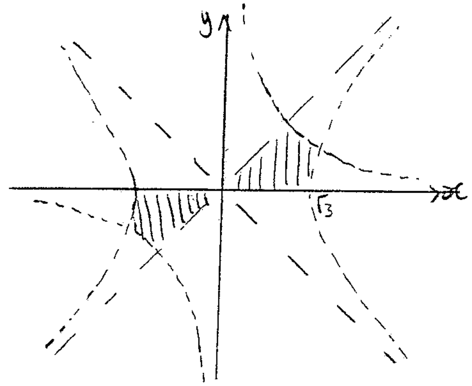


(ii) $Im(z^2) = 4$

ie $xy = 2$



(iii) note $x^2 - y^2 = 0$ is a pair of lines $y = \pm x$



(b) $\int_0^{\pi/3} \frac{dx}{1-\sin x} = \int_0^{\pi/3} \frac{1+\sin x}{1-\sin^2 x}\,dx$

$= \int_0^{\pi/3} \frac{1+\sin x}{\cos^2 x}\,dx$

$= \int_0^{\pi/3} \sec^2 x\,dx + \int_0^{\pi/3} \frac{\sin x}{\cos^2 x}\,dx$

$= \left[\tan x + \cos^{-1} x\right]_0^{\pi/3}$

$= \sqrt{3} + 2 - 1 = \sqrt{3} + 1$

(or 't' results see end)

(c) $P(2+i) = 0 \quad P(1-3i) = 0$

$\therefore P(2-i) = 0 \quad P(1+3i) = 0$

as coeff's real, roots are conjugates

$\therefore$ roots are $x_1 = 2+i \quad x_2 = 2-i$
$x_3 = 1-3i \quad x_4 = 1+3i$

sum of roots $= 6 = -b$

$\therefore b = -6$

product of roots $= e$

$= (2+i)(2-i)(1+3i)(1-3i)$

$= 50$

3 (a) LHS $= \operatorname{cosec} 2\theta + \cot 2\theta$

$= \frac{1}{\sin 2\theta} + \frac{\cos 2\theta}{\sin 2\theta}$

$= \frac{1+\cos 2\theta}{\sin 2\theta} = \frac{2\cos^2\theta}{2\cos\theta\sin\theta}$

$= \cot\theta$ = RHS QED

hence $\cot\frac{\pi}{8} = \operatorname{cosec}\frac{\pi}{4} + \cot\frac{\pi}{4}$

$= \sqrt{2} + 1$

& $\operatorname{cosec} 2\theta = \cot\theta - \cot 2\theta$

$\therefore \operatorname{cosec}\frac{2\pi}{15} = \cot\frac{\pi}{15} - \cot\frac{2\pi}{15}$

$\operatorname{cosec}\frac{4\pi}{15} = \cot\frac{2\pi}{15} - \cot\frac{4\pi}{15}$

$\operatorname{cosec}\frac{8\pi}{15} = \cot\frac{4\pi}{15} - \cot\frac{8\pi}{15}$

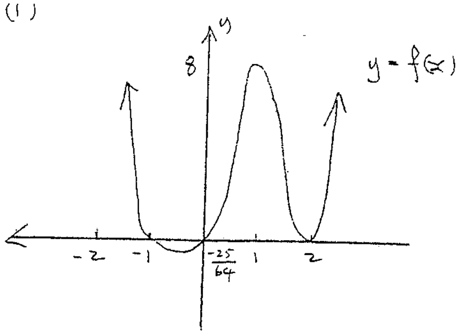
$\operatorname{cosec}\frac{16\pi}{15} = \cot\frac{8\pi}{15} - \cot\frac{16\pi}{15}$

$\therefore$ LHS $= \cot\frac{\pi}{15} - \cot\frac{16\pi}{15}$

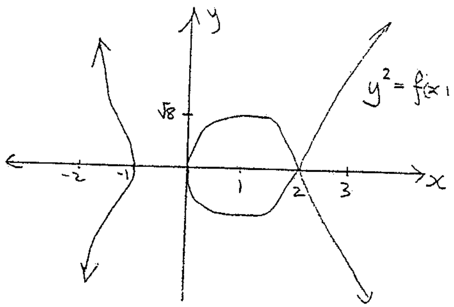
$= 0$ = RHS QED

as $\cot\frac{16\pi}{15} = \cot\frac{\pi}{15}$

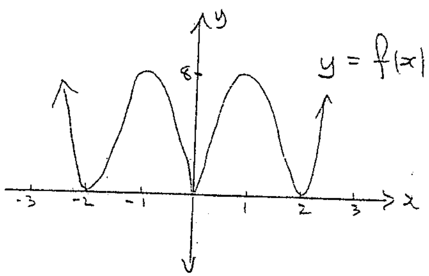
(b) (i)



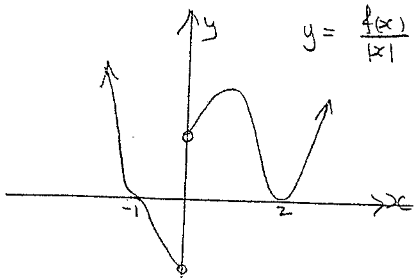
(ii)



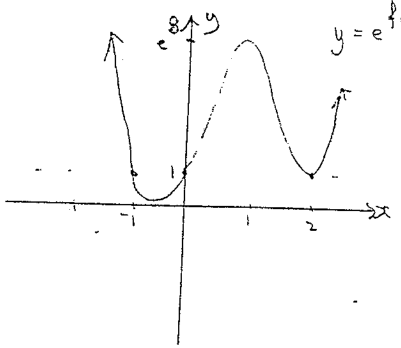
(iii)



(iv)



(v)



(4) (a) $4x^2+9y^2=36 \cap 4x^2-y^2=4$

$\therefore \quad 10y^2 = 32$

$y = \pm \cancel{\frac{2\sqrt2}{5}} \ \frac{4}{\sqrt5}$

$\therefore \quad x = \pm \cancel{\frac{3\sqrt2}{10}} \ \frac{\pm 3}{\sqrt5}$

Ellipse: $\quad 8x + 18y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-4x}{9y}$

m_{tan} at $\left(\frac{3\sqrt2}{10}, \frac{2\sqrt2}{5}\right) = -\frac{1}{3}$

Hyperbola: $\quad 8x - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{4x}{y}$

m_{tan} at $\left(\frac{3\sqrt2}{10}, \frac{2\sqrt2}{5}\right) = 3$

$\therefore \quad m_1 m_2 = -1 \quad \therefore \perp$

similarly for $(x,y) = \left(-\frac{3\sqrt2}{10}, -\frac{2\sqrt2}{5}\right)$ etc

b) $z = (1+ic)^6$

$= 1 + 6ic + 15i^2c^2 + 20i^3c^3 + 15i^4c^4 + 6i^5c^5 + i^6c^6$

for z real, $\text{Im}(z) = 0$

ie $z = 1 - 15c^2 + 15c^4 - c^6 + i(6c - 20c^3 + 6c^5)$

ie $6c - 20c^3 + 6c^5 = 0$

$2c(3 - 10c^2 + 3c^4) = 0$

$2c(3c^2-1)(c^2-3) = 0$

$c = 0, \quad c = \pm\frac{1}{\sqrt3}, \quad c = \pm\sqrt3$

c) $P\left(ct, \frac{c}{t}\right)$ on $xy = c^2$

$\frac{dy}{dx} = \frac{-c^2}{x^2}$

$m_{normal} = \frac{x^2}{c^2}$

eqn of normal is

$\left(y - \frac{c}{t}\right) = \frac{c^2t^2}{c^2}(x - ct)$

$ty - c = t^3(x - ct)$

$t^3x - ty = ct^4 - c$

now $t^3x - ty = ct^4 - c \cap xy = c^2$

ie $t^3x - \frac{tc^2}{x} = ct^4 - c$

$t^3x^2 - x(ct^4 - c) - tc^2 = 0$

$x = \frac{ct^4 - c \pm \sqrt{(ct^4-c)^2 + 4t^4c^2}}{2t^3}$

$\Delta = c^2t^8 - 2c^2t^4 + c^2 + 4t^4c^2$

$= c^2(t^8 + 2t^4 + 1)$

$= c^2(t^4+1)^2$

$\therefore \quad x_1 = \frac{ct^4 - c + ct^4 + c}{2t^3} = ct$

$x_2 = \frac{ct^4 - c - ct^4 - c}{2t^3} = \frac{-c}{t^3}$

$y_2 = \frac{c^2}{\frac{-c}{t^3}} = -ct^3$

cuts again at $\left(\frac{-c}{t^3}, -ct^3\right)$

(5) (a) $\int e^x \sin x \, dx$

$I = e^x \sin x - \int e^x \cos x \, dx$

$= e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx\right]$

$\therefore \quad 2I = e^x(\sin x - \cos x)$

$I = \frac{e^x(\sin x - \cos x)}{2}$

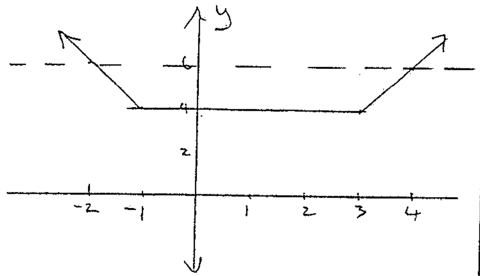
(b) $f(x) = |x-3| + |x+1|$

$x \geq 3 \quad f(x) = 2x - 2$

$x \leq -1 \quad f(x) = -2x + 2$

$-1 < x < 3 \quad f(x) = 4$

ie



ie $|x-3|+|x+1|>6$
for $x>4, \quad x<-2$.

(c)(i) $f(x)=\ln(\sec x+\tan x)$

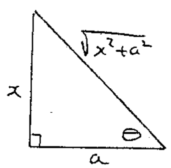
$f'(x)=\frac{1}{\sec x+\tan x}\cdot\left(\sec^2 x-\frac{\sin x}{-\cos x}\right)$

$=\frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)}$

$=\sec x$ QED

(ii) $\int\frac{dx}{\sqrt{a^2+x^2}}$ $\qquad x=a\tan\theta$, $dx=a\sec^2\theta\,d\theta$

$=\int\frac{a\sec^2\theta\,d\theta}{\sqrt{a^2+a^2\tan^2\theta}}$



$=\int\sec\theta\,d\theta$

$=\ln(\sec\theta+\tan\theta)+c_1$

$=\ln\left[\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right]+c_1$

$=\ln\sqrt{x^2+a^2}+x+c_2$

(d) $P(x)=x^4+x^3-3x^2-5x-2=0$

(i) let $X=1+\frac{1}{x}$

$x=\frac{1}{X-1}$

∴ eqn is

$\frac{1}{(x-1)^4}+\frac{1}{(x-1)^3}-\frac{3}{(x-1)^2}-\frac{5}{(x-1)}-2=0$

$1+(x-1)-3(x-1)^2-5(x-1)^3-2(x-1)^4$

$1+x-1-3x^2+6x-3-5x^3+15x^2-15x+5$

$-2x^4+8x^3-12x^2+8x-2=0$

$-2x^4+3x^3-2=0$

$2x^4-3x^3+2=0=Q(x)$

(ii) $P'(x)=4x^3+3x^2-6x-5$

$P''(x)=12x^2+6x-6$

$=6(x+1)(2x-1)$

triple root is either $x=-1$, or $x=\frac{1}{2}$

ie $x=-1$ on inspection

∴ $P(x)=(x+1)^3(x-a)$

$=(x+1)^3(x-2)$ on insp

6(a)(i) $P(\text{cis}\,\frac{\pi}{7})=(\text{cis}\,\frac{\pi}{7})^7+1$

$=\text{cis}\,\pi+1$

$=-1+1=0$

(ii) ie $w^1, w^3, w^5, w^7, \ldots$
is a zero

$w^3=\text{cis}\,\frac{3\pi}{7}$, $w^5=\frac{5\pi}{7}$ etc

∴ $P(w^3)=\text{cis}\,3\pi+1=0$

$P(w^5)=\text{cis}\,5\pi+1=0$

etc $P(w^n)=\text{cis}\,n\pi+1$ (n odd)

$=0$

(iii) $\frac{(z^7+1)}{(z+1)}=z^6-z^5+z^4-z^3+z^2-z+1$
on inspection

ie $(z^7+1)=(z+1)\,Q(z)$

as w is a zero of $P(z)$

& $w\neq-1$ w is a zero of $Q(z)$

(iv) $Q(z)$ has zeros

$w, w^3, w^5, w^9, w^{11}, w^{13}$

ie $\text{cis}\pm\frac{\pi}{7}, \text{cis}\pm\frac{3\pi}{7}, \text{cis}\pm\frac{5\pi}{7}$

(b)(i) $\cosh^2x=\left(\frac{e^x+e^{-x}}{2}\right)^2$

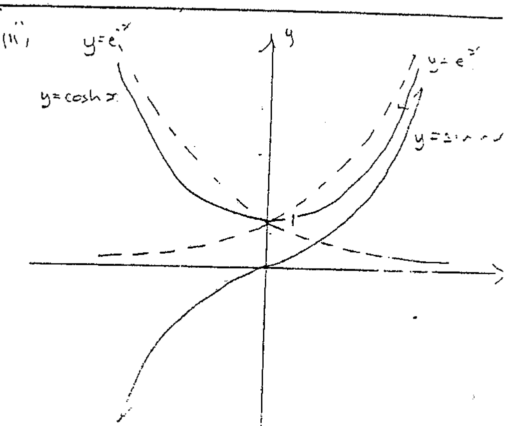
$=\frac{e^{2x}+2+e^{-2x}}{4}$

$\sinh^2x=\frac{e^{2x}-2+e^{-2x}}{4}$

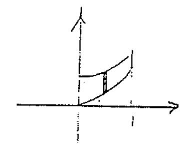
∴ $\cosh^2x-\sinh^2x=\frac{4}{4}=1$

QED

(ii)



(iii)



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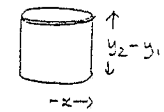


$\delta V=2\pi\int_0^1 x\cdot(y_2-y_1)\,dx$

∴ $V=2\pi\int_0^1 x(\cosh x-\sinh x)\,dx$

$\cosh x-\sinh x=\frac{e^x+e^{-x}-(e^x-e^{-x})}{2}$

$=e^{-x}$

$=2\pi\int_0^1 xe^{-x}\,dx$

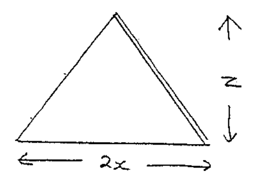
$v=x$ $\quad du=e^{-x}dx$
$dv=dx$ $\quad u=-e^{-x}$

$V=2\pi\left[-xe^{-x}+\int_0^1 e^{-x}dx\right]_0^1$

$=2\pi\left[-xe^{-x}-e^{-x}\right]_0^1=2\pi\left[-e^{-x}(x+1)\right]_0^1$

$=2\pi\left[-\frac{1}{e}\cdot2+1\cdot1\right]=2\pi\left(1-\frac{2}{e}\right)\,u^3$

7.(a)



$$\delta V = \tfrac{1}{2}\cdot 2x\cdot z\cdot \delta y = xz\,\delta y$$

and $\dfrac{x^2}{15^2} + \dfrac{y^2}{10^2} = 1$, $\dfrac{z^2}{15^2} + \dfrac{y^2}{10^2} = 1$

$$x^2 = 15^2\left(1 - \frac{y^2}{10^2}\right)$$

$$x = \frac{15}{10}\sqrt{100 - y^2}$$

sim. $z = \dfrac{15}{10}\sqrt{100-y^2}$

$$\therefore \quad \delta V = \frac{9}{4}(100-y^2)\,\delta y$$

$$V = \int_{-10}^{10} \frac{9}{4}(100-y^2)\,dy$$

$$= \frac{9}{2}\int_0^{10} 100 - y^2\,dy$$

$$= \frac{9}{2}\left[100y - \tfrac{1}{3}y^3\right]_0^{10}$$

$$= \frac{9}{2}\left[\left(1000 - \frac{1000}{3}\right) - 0\right]$$

$$= 3000 \text{ cm}^3$$

(b) (i) $\displaystyle\int \frac{x^3}{(x^2+1)^3}\,dx$

$u = x^2+1$
$x^2 = u - 1$
$du = 2x\,dx$

$$I = \frac{1}{2}\int \frac{(u-1)\,du}{u^3}$$

$$= \frac{1}{2}\int\left(\frac{1}{u^2} - \frac{1}{u^3}\right)du$$

$$= \frac{1}{2}\left(-u^{-1} + \tfrac{1}{2}u^{-2}\right) + c$$

$$= \frac{1}{4(x^2+1)^2} - \frac{1}{2(x^2+1)} + c$$

$$= \frac{1 - 2(x^2+1)}{4(x^2+1)^2} + c$$

$$= \frac{-(2x^2+1)}{4(x^2+1)^2} + c$$

(ii)

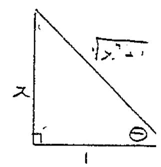
$x = \tan\theta$
$dx = \sec^2\theta\,d\theta$

$$I = \int \frac{\tan^3\theta\,\sec^2\theta}{(\tan^2+1)^3}\,d\theta$$

$$= \int \frac{\tan^3\theta}{\sec^4\theta}\,d\theta$$

$$= \int \sin^3\theta\cos\theta\,d\theta$$

$$= \frac{1}{4}\sin^4\theta + c$$



$$= \frac{x^4}{4(x^2+1)^2} + c$$

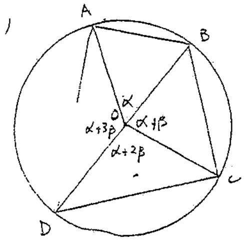
note: $\dfrac{x^4}{4(x^2+1)^2} - \dfrac{1}{4}$

$$= \frac{x^4 - (x^2+1)^2}{4(x^2+1)^2}$$

$$= \frac{-2x^2-1}{4(x^2+1)^2}$$

the integrals differ by $\frac{1}{4}$ (a constant)

8.(a)



(i) $\alpha + (\alpha+\beta) + (\alpha+2\beta) + (\alpha+3\beta) = 2\pi$

$4\alpha + 6\beta = 2\pi$

$\therefore \quad 2\alpha + 3\beta = \pi$ QED

$\therefore \quad D\hat{O}B = \pi \quad \therefore$ DB a diameter

$\therefore \quad D\hat{A}B = \frac{\pi}{2} = D\hat{C}B$

(ii) in $\triangle DAB$, $B\hat{D}A = \dfrac{\alpha}{2}$ ($\angle$ on circum)

in $\triangle BDC$, $B\hat{D}C = \dfrac{\alpha+\beta}{2}$

$\therefore \quad \sin\dfrac{\alpha}{2} = \dfrac{AB}{2r}$, $\cos\dfrac{\alpha}{2} = \dfrac{AD}{2r}$

Area $\triangle BAD = \frac{1}{2}\cdot AB\cdot AD$
$= 2r^2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$
$= r^2\sin\alpha$

Simil $BC = 2r\sin\dfrac{\alpha+\beta}{2}$, $CD = 2r\cos\dfrac{\alpha+\beta}{2}$

Area $\triangle BDC = 2r^2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2}$
$= r^2\sin(\alpha+\beta)$

$\therefore$ Area ABCD $= r^2(\sin\alpha + \sin(\alpha+\beta))$

(b) (i) Area of rectangles $\sqrt{1}+\sqrt{2}+\sqrt{3}$ is greater than ([illegible]) the area under the curve as $f(x)$ increasing $\forall x > 0$

$$\therefore \quad \sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{n} \geqslant \int_0^n \sqrt{x}\,dx$$

$$\int_0^n \sqrt{x}\,dx = \left[\tfrac{2}{3}\sqrt{x^3}\right]_0^n$$

$$= \frac{2}{3}n\sqrt{n}$$

ie $\sqrt{1}+\sqrt{2}+\ldots+\sqrt{n} \geqslant \displaystyle\int_0^n \sqrt{x}\,dx = \frac{2}{3}n\sqrt{n}$

(ii) $(4k+3)\sqrt{k} = \text{LHS}$

$(4k+3)^2 k = (\text{LHS})^2$

$= k(16k^2+24k+9) = 16k^3+24k^2+9k$

$(\text{RHS})^2 = (4k+1)^2(k+1)$
$= k(16k^2+8k+1) + 16k^2+8k+1$
$= 16k^3+24k^2+9k+1$

ie LHS < RHS

(iii) $n = 1$ LHS $= \sqrt{1} = 1$

RHS $= \dfrac{4+3}{6}\sqrt{1} = \dfrac{7}{6}$

true

assume $n = k$ true

ie $\sqrt{1}+\sqrt{2}+\ldots+\sqrt{k} \leqslant \dfrac{(4k+3)\sqrt{k}}{6}$

RTP: $\sqrt{1}+\sqrt{2}+\ldots+\sqrt{k}+\sqrt{k+1} \leqslant \dfrac{(4(k+1)+3)\sqrt{k+1}}{6} = \dfrac{(4k+7)\sqrt{k+1}}{6}$

now $\sqrt{1}+\sqrt{2}+\ldots\sqrt{k}+\sqrt{k+1} \leqslant \dfrac{(4k+3)}{6}\sqrt{k}+\sqrt{k+1}$

$+ \quad \dfrac{(4k+3)}{6}\sqrt{k} + \sqrt{k+1} < \dfrac{(4k+1)}{6}\sqrt{k+1} + \sqrt{k+1}$ from (ii)

$= \dfrac{(4k+7)\sqrt{k+1}}{6}$

ie $\sqrt{1}+\sqrt{2}+\ldots+\sqrt{k+1} \leqslant \dfrac{(4k+7)\sqrt{k+1}}{6}$

1996 4unit TRIAL SOLUTIONS

1 (a) (i) $\int \tan^3 x\,dx$

$= \int (\sec^2 x - 1)\tan x\,dx$

$= \int \sec^2 x \tan x\,dx - \int \frac{\sin x}{\cos x}\,dx$

$= \frac{1}{2}\tan^2 x + \ln\cos x + c$

(ii) $\int \frac{e^x + e^{2x}}{1+e^{2x}}\,dx$

$= \int \frac{e^x\,dx}{1+e^{2x}} + \int \frac{e^{2x}\,dx}{1+e^{2x}}$

$= \tan^{-1} e^x + \frac{1}{2}\ln(1+e^{2x}) + c$

(b) (i) $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

$= \int_{-1}^{1} \frac{dx}{(x+1)^2+4} = \frac{1}{2}\left[\tan^{-1}\frac{x+1}{2}\right]_{-1}^{1}$

$= \frac{\pi}{4}\cdot\frac{1}{2} = \frac{\pi}{8}$

(ii) $\int_4^{12} \frac{dx}{(4+x)\sqrt{x}}$

$= \int_2^{2\sqrt{3}} \frac{2\,du}{4+u^2}$

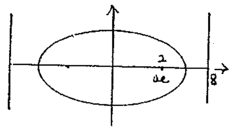
$u = \sqrt{x}$
$du = \frac{1}{2\sqrt{x}}\,dx$
$x=4 \quad u=2$
$x=12 \quad u=2\sqrt{3}$

$= \left[\tan^{-1}\frac{u}{2}\right]_2^{2\sqrt{3}}$

$= (\tan^{-1}\sqrt{3} - \tan^{-1} 1)$

$= \left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{12}$

(c)



$\frac{a}{e} = 8$, $ae = 2$ $\left(e = \frac{2}{a}\right)$

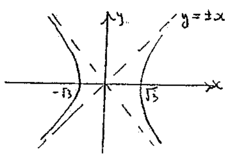
$\therefore a^2 = 16 \quad a = 4$
$e = \frac{1}{2}$

$\therefore b^2 = 16\left(1 - \frac{1}{4}\right) = 12$

ie $\frac{x^2}{16} + \frac{y^2}{12} = 1$ is the eqn.

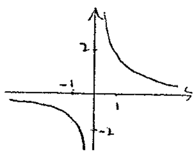
2 (i) $z^2 = x^2 + 2xyi - y^2$

ie $x^2 - y^2 = 3$ is $Re(z^2) = 3$

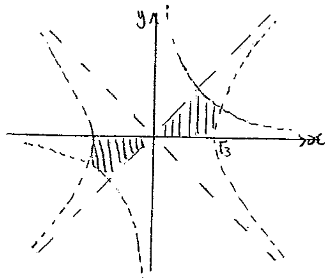


(ii) $Im(z^2) = 4$

ie $xy = 2$



(iii) note $x^2 - y^2 = 0$ is a pair of lines $y = \pm x$



(b) $\int_0^{\frac{\pi}{3}} \frac{dx}{1-\sin x} = \int_0^{\frac{\pi}{3}} \frac{1+\sin x}{1-\sin^2 x}\,dx$

$= \int_0^{\frac{\pi}{3}} \frac{1+\sin x}{\cos^2 x}\,dx$

$= \int_0^{\frac{\pi}{3}} \sec^2 x\,dx + \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos^2 x}\,dx$

$= \left[\tan x + \cos^{-1} x\right]_0^{\frac{\pi}{3}}$

$= \sqrt{3} + 2 - 1 = \sqrt{3} + 1$

(or 't' results see end)

(c) $P(2+i) = 0 \quad P(1-3i) = 0$

$\therefore P(2-i) = 0 \quad P(1+3i) = 0$

as coeff's real, roots are conjugates

$\therefore$ roots are $x_1 = 2+i \quad x_2 = 2-i$
$x_3 = 1-3i \quad x_4 = 1+3i$

sum of roots $= 6 = -b$
$\therefore b = -6$

product of roots $= e$
$= (2+i)(2-i)(1+3i)(1-3i)$
$= 50$

3 (a) LHS $= \operatorname{cosec} 2\theta + \cot 2\theta$

$= \frac{1}{\sin 2\theta} + \frac{\cos 2\theta}{\sin 2\theta}$

$= \frac{1+\cos 2\theta}{\sin 2\theta} = \frac{2\cos^2\theta}{2\cos\theta\sin\theta}$

$= \cot\theta = $ RHS Q

hence $\cot\frac{\pi}{8} = \operatorname{cosec}\frac{\pi}{4} + c$

$= \sqrt{2} + 1$

& $\operatorname{cosec} 2\theta = \cot\theta - \cot 2\theta$

$\therefore \operatorname{cosec}\frac{2\pi}{15} = \cot\frac{\pi}{15} - \cot\frac{2\pi}{15}$

$\operatorname{cosec}\frac{4\pi}{15} = \cot\frac{2\pi}{15} - \cot\frac{4\pi}{15}$

$\operatorname{cosec}\frac{8\pi}{15} = \cot\frac{4\pi}{15} - \cot\frac{8\pi}{15}$

$\operatorname{cosec}\frac{16\pi}{15} = \cot\frac{8\pi}{15} - \cot\frac{16\pi}{15}$

$\therefore$ LHS $= \cot\frac{\pi}{15} - \cot\frac{16\pi}{15}$

$= 0 = $ RHS QED

as $\cot\frac{16\pi}{15} = \cot\frac{\pi}{15}$

$\therefore$ by principles of MI etc.

(iv) now

$\frac{2}{3}n\sqrt{n} \leqslant \sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n} \leqslant \frac{4n+3}{6}\sqrt{n}$

ie

$\frac{2}{3}.10000.100 \leqslant \sqrt{1} + \sqrt{2} + \ldots \sqrt{10000} \leqslant \frac{40003}{6}.100$

$666667 \leqslant$ ——— $\leqslant 666717 \approx$

ie $\sqrt{1} + \sqrt{2} + \ldots + \sqrt{10000} \doteqdot 666700$

2 (b) t results

$t = \tan\frac{x}{2}$
$\sin x = \frac{2t}{1+t^2}$
$t = 0$ when $x = 0$
$t = \frac{1}{\sqrt{3}}$ when $x = \frac{\pi}{3}$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

$I = \int_0^{\frac{1}{\sqrt{3}}} \frac{2\,dt}{1 - \frac{2t}{1+t^2}} \cdot \frac{1}{1+t^2}$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2\,dt}{(t-1)^2} = -2\left[(t-1)^{-1}\right]_0^{\frac{1}{\sqrt{3}}}$

$= \sqrt{3} + 1$